

SO(4) Re-revisited

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To the memory of Steve Jobs ¹

Abstract

In this note an explicit expression of $\exp A$ ($A \in \mathfrak{so}(4)$) is given in terms of the magic matrix by Makhlin.

1 Introduction

The four-dimensional special orthogonal group $SO(4)$ plays an important role in both Mathematics and Physics, and maybe in Chemistry. Since it is semi-simple, essential properties reduce to those of $SU(2)$ or $SO(3)$ ($\cong SU(2)/\mathbf{Z}_2$). See [1] for a comprehensive introduction to this topic and we recommend [2] as a good text-book of Group and Topology.

Let us recall the definition:

$$SO(4) = \{O \in M(4; \mathbf{R}) \mid O^t O = O O^t = I_4, \det O = 1\}, \quad (1)$$

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¹Steven Paul Jobs (1955–2011)

where t denotes the transpose, I_4 the four dimensional unit matrix and $\det O$ the determinant of O . Its Lie algebra is given by

$$so(4) = \{A \in M(4; \mathbf{R}) \mid A^t = -A, \text{tr} A = 0\}. \quad (2)$$

Then it is well-known that every element of $SO(4)$ can be written as

$$SO(4) = \{e^A \mid A \in so(4)\} \quad (3)$$

where $e^A (= \exp A)$ is the exponential defined by

$$e^A = I_4 + A + \frac{1}{2!}A^2 + \cdots + \frac{1}{n!}A^n + \cdots. \quad (4)$$

The form is simple and beautiful, while to calculate e^A is another problem. In fact, it is very difficult and its explicit form has not been reported as far as we know. In this note we re-revisit this problem and give a “super smart” form (see the concluding remarks) to e^A in terms of the magic matrix by Makhlin.

Let us set up the problem once more. Calculate the exponential

$$e^A ; A = \begin{pmatrix} 0 & f_{12} & f_{13} & f_{14} \\ -f_{12} & 0 & f_{23} & f_{24} \\ -f_{13} & -f_{23} & 0 & f_{34} \\ -f_{14} & -f_{24} & -f_{34} & 0 \end{pmatrix} \in so(4) \quad (5)$$

explicitly. For this purpose we need the magic matrix by Makhlin [3].

2 Magic Matrix

In this section we review the result in [4], [5] within our necessity, which is a bit different from the original in [3]. This section is also a brief introduction to **Quantum Computation** for undergraduates.

The 1-qubit space is the two dimensional vector space over \mathbf{C} , namely

$$\mathbf{C}^2 = \text{Vect}_{\mathbf{C}}\{|0\rangle, |1\rangle\} \equiv \{\alpha|0\rangle + \beta|1\rangle \mid \alpha, \beta \in \mathbf{C}\},$$

where

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (6)$$

Let $\{\sigma_1, \sigma_2, \sigma_3\}$ be the Pauli matrices acting on the space

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (7)$$

and we denote by 1_2 the two dimensional unit matrix

$$1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (8)$$

Next let us consider the 2-qubit space. We start with the Kronecker product $a \otimes b$ of two \mathbf{C}^2 vectors a and b :

$$a \otimes b \equiv \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \equiv \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix}.$$

They form a set of tensor products:

$$\mathbf{C}^2 \otimes \mathbf{C}^2 = \{a \otimes b \mid a, b \in \mathbf{C}^2\}.$$

The 2-qubit space is a vector space generated by them:

$$\mathbf{C}^2 \hat{\otimes} \mathbf{C}^2 = \left\{ \sum_{j=1}^k \lambda_j a_j \otimes b_j \mid a_j, b_j \in \mathbf{C}^2, \lambda_j \in \mathbf{C}, k \in \mathbf{N} \right\} \cong \mathbf{C}^4.$$

Then we have

$$\mathbf{C}^2 \hat{\otimes} \mathbf{C}^2 = \text{Vect}_{\mathbf{C}}\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$$

where $|ij\rangle = |i\rangle \otimes |j\rangle$ ($i, j \in \{0, 1\}$) for simplicity.

For $A, B \in M(2; \mathbf{C})$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

the Kronecker product $A \otimes B$ is defined by

$$(A \otimes B)(a \otimes b) = (Aa) \otimes (Bb).$$

Its explicit form is given by

$$\begin{aligned} A \otimes B &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}. \end{aligned}$$

Readers should check it. From this we have the multiplication

$$(A_1 \otimes B_1)(A_2 \otimes B_2) = A_1 A_2 \otimes B_1 B_2$$

for $A_1, A_2, B_1, B_2 \in M(2; \mathbf{C})$.

By $H_0(2; \mathbf{C})$ we denote the set of all traceless hermitian matrices in $M(2; \mathbf{C})$. Then it is well-known

$$H_0(2; \mathbf{C}) = \{a \equiv a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 \mid a_1, a_2, a_3 \in \mathbf{R}\}$$

and $H_0(2; \mathbf{C}) \cong su(2)$ where $su(2)$ is the Lie algebra of the group $SU(2)$

$$su(2) = \{i(a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3) \mid a_1, a_2, a_3 \in \mathbf{R}\}.$$

The (famous) Bell bases $\{|\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle, |\Psi_4\rangle\}$ in $\mathbf{C}^2 \hat{\otimes} \mathbf{C}^2$ are defined by

$$\begin{aligned} |\Psi_1\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), & |\Psi_2\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \\ |\Psi_3\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), & |\Psi_4\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \end{aligned} \tag{9}$$

and by making use of them we can give the isomorphism $(SU(2) \otimes SU(2) \cong SO(4))$ as the adjoint action (the Makhlin's theorem) as follows

$$F : SU(2) \otimes SU(2) \longrightarrow SO(4), \quad F(P \otimes Q) = R^\dagger(P \otimes Q)R$$

where

$$R = (|\Psi_1\rangle, -i|\Psi_2\rangle, -|\Psi_3\rangle, -i|\Psi_4\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & -i & -1 & 0 \\ 0 & -i & 1 & 0 \\ 1 & 0 & 0 & i \end{pmatrix}. \quad (10)$$

Note that the unitary matrix R is a bit different from Q in [3].

Let us consider this problem at the Lie algebra level because it is in general not easy to treat it directly at the Lie group level:

$$\begin{array}{ccc} \mathfrak{L}(SU(2) \otimes SU(2)) & \xrightarrow{f} & \mathfrak{L}(SO(4)) \\ \exp \downarrow & & \downarrow \exp \\ SU(2) \otimes SU(2) & \xrightarrow{F} & SO(4) \end{array}.$$

Since the Lie algebra of $SU(2) \otimes SU(2)$ is

$$\mathfrak{L}(SU(2) \otimes SU(2)) = \{i(a \otimes 1_2 + 1_2 \otimes b) \mid a, b \in H_0(2; \mathbf{C})\},$$

we have only to examine

$$f(i(a \otimes 1_2 + 1_2 \otimes b)) = iR^\dagger(a \otimes 1_2 + 1_2 \otimes b)R \in \mathfrak{L}(SO(4)) \equiv \mathfrak{so}(4). \quad (11)$$

For $a = \sum_{j=1}^3 a_j \sigma_j$ and $b = \sum_{j=1}^3 b_j \sigma_j$ some algebra gives

$$iR^\dagger(a \otimes 1_2 + 1_2 \otimes b)R = \begin{pmatrix} 0 & a_1 + b_1 & a_2 - b_2 & a_3 + b_3 \\ -(a_1 + b_1) & 0 & a_3 - b_3 & -(a_2 + b_2) \\ -(a_2 - b_2) & -(a_3 - b_3) & 0 & a_1 - b_1 \\ -(a_3 + b_3) & a_2 + b_2 & -(a_1 - b_1) & 0 \end{pmatrix}. \quad (12)$$

Conversely, for

$$A = \begin{pmatrix} 0 & f_{12} & f_{13} & f_{14} \\ -f_{12} & 0 & f_{23} & f_{24} \\ -f_{13} & -f_{23} & 0 & f_{34} \\ -f_{14} & -f_{24} & -f_{34} & 0 \end{pmatrix} \in \mathfrak{so}(4)$$

some algebra gives

$$RAR^\dagger = i(a \otimes 1_2 + 1_2 \otimes b) \quad (13)$$

with

$$a = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 = \frac{f_{12} + f_{34}}{2}\sigma_1 + \frac{f_{13} - f_{24}}{2}\sigma_2 + \frac{f_{14} + f_{23}}{2}\sigma_3, \quad (14)$$

$$b = b_1\sigma_1 + b_2\sigma_2 + b_3\sigma_3 = \frac{f_{12} - f_{34}}{2}\sigma_1 - \frac{f_{13} + f_{24}}{2}\sigma_2 + \frac{f_{14} - f_{23}}{2}\sigma_3, \quad (15)$$

$$\begin{aligned} a_1 &= \frac{f_{12} + f_{34}}{2}, & a_2 &= \frac{f_{13} - f_{24}}{2}, & a_3 &= \frac{f_{14} + f_{23}}{2}, \\ b_1 &= \frac{f_{12} - f_{34}}{2}, & b_2 &= -\frac{f_{13} + f_{24}}{2}, & b_3 &= \frac{f_{14} - f_{23}}{2}. \end{aligned}$$

Readers should check it.

Comment It is very interesting to note that a is the **self-dual** part and b the **anti-self-dual** one. See [6] for details.

Last, we list a well-known formula for the exponentiation of $su(2)$ for later convenience.

For

$$X = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3, \quad |X| \stackrel{\text{def}}{=} \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad x_j \in \mathbf{R},$$

it is easy to obtain the following

$$\begin{aligned} e^{iX} &= \cos |X| 1_2 + \frac{\sin |X|}{|X|} iX \\ &= \begin{pmatrix} \cos |X| + \frac{\sin |X|}{|X|} ix_3 & \frac{\sin |X|}{|X|} i(x_1 - ix_2) \\ \frac{\sin |X|}{|X|} i(x_1 + ix_2) & \cos |X| - \frac{\sin |X|}{|X|} ix_3 \end{pmatrix}. \end{aligned} \quad (16)$$

From this and (14), (15) we have

$$e^{ia} = \begin{pmatrix} \cos |a| + \frac{\sin |a|}{|a|} ia_3 & \frac{\sin |a|}{|a|} i(a_1 - ia_2) \\ \frac{\sin |a|}{|a|} i(a_1 + ia_2) & \cos |a| - \frac{\sin |a|}{|a|} ia_3 \end{pmatrix}, \quad (17)$$

$$e^{ib} = \begin{pmatrix} \cos |b| + \frac{\sin |b|}{|b|} ib_3 & \frac{\sin |b|}{|b|} i(b_1 - ib_2) \\ \frac{\sin |b|}{|b|} i(b_1 + ib_2) & \cos |b| - \frac{\sin |b|}{|b|} ib_3 \end{pmatrix}, \quad (18)$$

where

$$|a| \stackrel{\text{def}}{=} \sqrt{a_1^2 + a_2^2 + a_3^2}, \quad |b| \stackrel{\text{def}}{=} \sqrt{b_1^2 + b_2^2 + b_3^2}.$$

3 Exact Calculation of the Exponential

The purpose of this section is to calculate e^A for

$$A = \begin{pmatrix} 0 & f_{12} & f_{13} & f_{14} \\ -f_{12} & 0 & f_{23} & f_{24} \\ -f_{13} & -f_{23} & 0 & f_{34} \\ -f_{14} & -f_{24} & -f_{34} & 0 \end{pmatrix} \in so(4).$$

First, we have

$$\begin{aligned} e^A &= I_4 e^A I_4 \\ &= R^{-1} R e^A R^{-1} R \quad (\Leftarrow R^\dagger = R^{-1}) \\ &= R^{-1} e^{R A R^{-1}} R \quad (\Leftarrow (13)) \\ &= R^\dagger e^{i(a \otimes 1_2 + 1_2 \otimes b)} R \\ &= R^\dagger e^{i(a \otimes 1_2)} e^{i(1_2 \otimes b)} R \\ &= R^\dagger (e^{ia} \otimes 1_2) (1_2 \otimes e^{ib}) R \\ &= R^\dagger (e^{ia} \otimes e^{ib}) R. \end{aligned} \tag{19}$$

Second, in order to write down $e^{ia} \otimes e^{ib}$ we set

$$T \equiv e^{ia} \otimes e^{ib} = (t_{ij}). \tag{20}$$

Then (17) and (18) give

$$\begin{aligned} t_{11} &= \left(\cos |a| + \frac{\sin |a|}{|a|} i a_3 \right) \left(\cos |b| + \frac{\sin |b|}{|b|} i b_3 \right), \\ t_{21} &= \left(\cos |a| + \frac{\sin |a|}{|a|} i a_3 \right) \frac{\sin |b|}{|b|} i (b_1 + i b_2), \\ t_{31} &= \frac{\sin |a|}{|a|} i (a_1 + i a_2) \left(\cos |b| + \frac{\sin |b|}{|b|} i b_3 \right), \\ t_{41} &= \frac{\sin |a|}{|a|} i (a_1 + i a_2) \frac{\sin |b|}{|b|} i (b_1 + i b_2); \end{aligned}$$

$$\begin{aligned}
t_{12} &= \left(\cos |a| + \frac{\sin |a|}{|a|} i a_3 \right) \frac{\sin |b|}{|b|} i (b_1 - i b_2), \\
t_{22} &= \left(\cos |a| + \frac{\sin |a|}{|a|} i a_3 \right) \left(\cos |b| - \frac{\sin |b|}{|b|} i b_3 \right), \\
t_{32} &= \frac{\sin |a|}{|a|} i (a_1 + i a_2) \frac{\sin |b|}{|b|} i (b_1 - i b_2), \\
t_{42} &= \frac{\sin |a|}{|a|} i (a_1 + i a_2) \left(\cos |b| - \frac{\sin |b|}{|b|} i b_3 \right); \\
\\
t_{13} &= \frac{\sin |a|}{|a|} i (a_1 - i a_2) \left(\cos |b| + \frac{\sin |b|}{|b|} i b_3 \right), \\
t_{23} &= \frac{\sin |a|}{|a|} i (a_1 - i a_2) \frac{\sin |b|}{|b|} i (b_1 + i b_2), \\
t_{33} &= \left(\cos |a| - \frac{\sin |a|}{|a|} i a_3 \right) \left(\cos |b| + \frac{\sin |b|}{|b|} i b_3 \right), \\
t_{43} &= \left(\cos |a| - \frac{\sin |a|}{|a|} i a_3 \right) \frac{\sin |b|}{|b|} i (b_1 + i b_2); \\
\\
t_{14} &= \frac{\sin |a|}{|a|} i (a_1 - i a_2) \frac{\sin |b|}{|b|} i (b_1 - i b_2), \\
t_{24} &= \frac{\sin |a|}{|a|} i (a_1 - i a_2) \left(\cos |b| - \frac{\sin |b|}{|b|} i b_3 \right), \\
t_{34} &= \left(\cos |a| - \frac{\sin |a|}{|a|} i a_3 \right) \frac{\sin |b|}{|b|} i (b_1 - i b_2), \\
t_{44} &= \left(\cos |a| - \frac{\sin |a|}{|a|} i a_3 \right) \left(\cos |b| - \frac{\sin |b|}{|b|} i b_3 \right).
\end{aligned}$$

Third, let us write down $R^\dagger T R$. Some algebra gives

$$\begin{aligned}
& R^\dagger T R \\
&= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & i & i & 0 \\ 0 & -1 & 1 & 0 \\ i & 0 & 0 & -i \end{pmatrix} \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & -i & -1 & 0 \\ 0 & -i & 1 & 0 \\ 1 & 0 & 0 & i \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} t_{11} + t_{41} + t_{14} + t_{44} & -i(t_{12} + t_{42} + t_{13} + t_{43}) & -t_{12} - t_{42} + t_{13} + t_{43} & i(-t_{11} - t_{41} + t_{14} + t_{44}) \\ i(t_{21} + t_{31} + t_{24} + t_{34}) & t_{22} + t_{32} + t_{23} + t_{33} & i(-t_{22} - t_{32} + t_{23} + t_{33}) & t_{21} + t_{31} - t_{24} - t_{34} \\ -t_{21} + t_{31} - t_{24} + t_{34} & i(t_{22} - t_{32} + t_{23} - t_{33}) & t_{22} - t_{32} - t_{23} + t_{33} & i(t_{21} - t_{31} - t_{24} + t_{34}) \\ i(t_{11} - t_{41} + t_{14} - t_{44}) & t_{12} - t_{42} + t_{13} - t_{43} & i(-t_{12} + t_{42} + t_{13} - t_{43}) & t_{11} - t_{41} - t_{14} + t_{44} \end{pmatrix}.
\end{aligned}$$

(21)

Last, we calculate each component above by making use of (20). If we set

$$e^A = R^\dagger T R = X = (x_{ij}) \quad (22)$$

then straightforward but long algebra gives

$$\begin{aligned}
x_{11} &= \frac{1}{2}(t_{11} + t_{41} + t_{14} + t_{44}) \\
&= \cos |a| \cos |b| - \frac{\sin |a| \sin |b|}{|a||b|}(a_1 b_1 - a_2 b_2 + a_3 b_3), \\
x_{21} &= \frac{i}{2}(t_{21} + t_{31} + t_{24} + t_{34}) \\
&= -\cos |a| \frac{\sin |b|}{|b|} b_1 - \frac{\sin |a|}{|a|} \cos |b| a_1 + \frac{\sin |a| \sin |b|}{|a||b|}(a_2 b_3 + a_3 b_2), \\
x_{31} &= \frac{1}{2}(-t_{21} + t_{31} - t_{24} + t_{34}) \\
&= \cos |a| \frac{\sin |b|}{|b|} b_2 - \frac{\sin |a|}{|a|} \cos |b| a_2 - \frac{\sin |a| \sin |b|}{|a||b|}(a_1 b_3 - a_3 b_1), \\
x_{41} &= \frac{i}{2}(t_{11} - t_{41} + t_{14} - t_{44}) \\
&= -\cos |a| \frac{\sin |b|}{|b|} b_3 - \frac{\sin |a|}{|a|} \cos |b| a_3 - \frac{\sin |a| \sin |b|}{|a||b|}(a_1 b_2 + a_2 b_1); \\
\\
x_{12} &= \frac{-i}{2}(t_{12} + t_{42} + t_{13} + t_{43}) \\
&= \cos |a| \frac{\sin |b|}{|b|} b_1 + \frac{\sin |a|}{|a|} \cos |b| a_1 + \frac{\sin |a| \sin |b|}{|a||b|}(a_2 b_3 + a_3 b_2), \\
x_{22} &= \frac{1}{2}(t_{22} + t_{32} + t_{23} + t_{33}) \\
&= \cos |a| \cos |b| - \frac{\sin |a| \sin |b|}{|a||b|}(a_1 b_1 + a_2 b_2 - a_3 b_3), \\
x_{32} &= \frac{i}{2}(t_{22} - t_{32} + t_{23} - t_{33}) \\
&= \cos |a| \frac{\sin |b|}{|b|} b_3 - \frac{\sin |a|}{|a|} \cos |b| a_3 + \frac{\sin |a| \sin |b|}{|a||b|}(a_1 b_2 - a_2 b_1), \\
x_{42} &= \frac{1}{2}(t_{12} - t_{42} + t_{13} - t_{43}) \\
&= \cos |a| \frac{\sin |b|}{|b|} b_2 + \frac{\sin |a|}{|a|} \cos |b| a_2 - \frac{\sin |a| \sin |b|}{|a||b|}(a_1 b_3 + a_3 b_1);
\end{aligned}$$

$$\begin{aligned}
x_{13} &= \frac{1}{2}(-t_{12} - t_{42} + t_{13} + t_{43}) \\
&= -\cos |a| \frac{\sin |b|}{|b|} b_2 + \frac{\sin |a|}{|a|} \cos |b| a_2 - \frac{\sin |a| \sin |b|}{|a||b|} (a_1 b_3 - a_3 b_1), \\
x_{23} &= \frac{i}{2}(-t_{22} - t_{32} + t_{23} + t_{33}) \\
&= -\cos |a| \frac{\sin |b|}{|b|} b_3 + \frac{\sin |a|}{|a|} \cos |b| a_3 + \frac{\sin |a| \sin |b|}{|a||b|} (a_1 b_2 - a_2 b_1), \\
x_{33} &= \frac{1}{2}(t_{22} - t_{32} - t_{23} + t_{33}) \\
&= \cos |a| \cos |b| + \frac{\sin |a| \sin |b|}{|a||b|} (a_1 b_1 + a_2 b_2 + a_3 b_3), \\
x_{43} &= \frac{i}{2}(-t_{12} + t_{42} + t_{13} - t_{43}) \\
&= \cos |a| \frac{\sin |b|}{|b|} b_1 - \frac{\sin |a|}{|a|} \cos |b| a_1 - \frac{\sin |a| \sin |b|}{|a||b|} (a_2 b_3 - a_3 b_2); \\
\\
x_{14} &= \frac{i}{2}(-t_{11} - t_{41} + t_{14} + t_{44}) \\
&= \cos |a| \frac{\sin |b|}{|b|} b_3 + \frac{\sin |a|}{|a|} \cos |b| a_3 - \frac{\sin |a| \sin |b|}{|a||b|} (a_1 b_2 + a_2 b_1), \\
x_{24} &= \frac{1}{2}(t_{21} + t_{31} - t_{24} - t_{34}) \\
&= -\cos |a| \frac{\sin |b|}{|b|} b_2 - \frac{\sin |a|}{|a|} \cos |b| a_2 - \frac{\sin |a| \sin |b|}{|a||b|} (a_1 b_3 + a_3 b_1), \\
x_{34} &= \frac{i}{2}(t_{21} - t_{31} - t_{24} + t_{34}) \\
&= -\cos |a| \frac{\sin |b|}{|b|} b_1 + \frac{\sin |a|}{|a|} \cos |b| a_1 - \frac{\sin |a| \sin |b|}{|a||b|} (a_2 b_3 - a_3 b_2), \\
x_{44} &= \frac{1}{2}(t_{11} - t_{41} - t_{14} + t_{44}) \\
&= \cos |a| \cos |b| + \frac{\sin |a| \sin |b|}{|a||b|} (a_1 b_1 - a_2 b_2 - a_3 b_3).
\end{aligned}$$

This completes the calculation of e^A ($A \in so(4)$). The form is of course not simple, while it is not so ugly in the Dirac sense. It is straightforward to verify the orthogonality $X^t X = X X^t = I_4$ and the unit determinant $\det X = 1$. See [7], [8], [9] for related topics.

Comment The matrix R is called the magic one by Makhlin. Readers must understand the reason why it is called **magic** through this paper.

Is it possible to calculate e^A for $A \in so(n)$ ($n \geq 5$) ? Unfortunately or fortunately, it has not been done at the present time, so we leave the problem to young readers.

Last, let us present an exercise to readers.

Exercise The three-dimensional special orthogonal group $SO(3)$ ($\cong SU(2)/\mathbf{Z}_2$) is a subgroup of $SO(4)$ and can be embedded into $SO(4)$ like

$$SO(3) \longrightarrow SO(4) : O \longmapsto \begin{pmatrix} O & \\ & 1 \end{pmatrix}.$$

Therefore, for $B \in so(3)$

$$B = \begin{pmatrix} 0 & a & c \\ -a & 0 & b \\ -c & -b & 0 \end{pmatrix} \quad (23)$$

we can write

$$\begin{pmatrix} e^B & \\ & 1 \end{pmatrix} = \exp \left\{ \begin{pmatrix} B & \\ & 0 \end{pmatrix} \right\} = \exp \left\{ \begin{pmatrix} 0 & a & c & 0 \\ -a & 0 & b & 0 \\ -c & -b & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}. \quad (24)$$

Since

$$\begin{pmatrix} 0 & a & c & 0 \\ -a & 0 & b & 0 \\ -c & -b & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in so(4)$$

we can easily calculate (24) and obtain e^B from the result in the preceding section.

Carry out this procedure.

4 Concluding Remarks

In this paper we wrote down $\exp A$ ($A \in so(4)$) explicitly. Its form is relatively simple and beautiful, and has not been given as far as we know.

While preparing this paper, a very sad news of Steve Jobs' death arrived. With grief I would like to dedicate this paper to his memory. His way of thinking is in my opinion based on three words

(1) simple, (2) easy to use, (3) beautiful.

These may be unified as “super smart”. It must be very important in almost all fields. Whether the result in the paper is “super smart” or not will be left to readers.

He also says

Stay hungry, Stay foolish.

That must be the spirit of Apple or Silicon Valley, and the spirit is strongly required by not only USA but also Japan.

We conclude this paper with the recent hot topic ... the OPERA experiment results [10]. Namely, **the speed of neutrino may exceed that of light in vacuum**. The result is “foolish” enough. At the present time it is impossible to conclude the superluminal neutrinos, so we are looking forward to the “super smart” interpretation of [10]. See also [11] and its references for a possibility.

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